Laplacian eigenmodes for the three sphere

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Abstract

The vector space $\mathcal{V}^k$ of the eigenfunctions of the Laplacian on the three sphere S^3, corresponding to the same eigenvalue $\lambda_k = -k\ (k+2)$, has dimension $(k+1)^2$. After recalling the standard bases for $\mathcal{V}^k$, we introduce a new basis B3, constructed from the reductions to S^3 of peculiar homogeneous harmonic polynomia involving null vectors. We give the transformation laws between this basis and the usual hyper-spherical harmonics.

Thanks to the quaternionic representations of S^3 and SO(4), we are able to write explicitely the transformation properties of B3, and thus of any eigenmode, under an arbitrary rotation of SO(4). This offers the possibility to select those functions of $\mathcal{V}^k$ which remain invariant under a chosen rotation of SO(4). When the rotation is an holonomy transformation of a spherical space S^3/Γ, this gives a method to calculates the eigenmodes of S^3/Γ, which remains an open probleme in general. We illustrate our method by (re-)deriving the eigenmodes of lens and prism space. In a forthcoming paper, we present the derivation for dodecahedral space.

1 Introduction

The eigenvalues of the Laplacian Δ of S^3 are of the form $\lambda_k = -k\ (k+2)$, where $k \in \mathbb{N}^+$. For a given value of k, they span the eigenspace $\mathcal{V}^k$ of dimension $(k+1)^2$. This vector space constitutes the $(k+1)^2$ dimensional irreductible representation of SO(4), the isometry group of S^3.

There are two commonly used bases (hereafter B1 and B2) for $\mathcal{V}^k$ which generalize in some sense (see below) the usual spherical harmonics $Y_{\ell m}$ for the two-sphere. The functions of these bases have a friendly behavior under some of the rotations of SO(4); this generalizes the property of the $Y_{\ell m}$ to be eigenfunctions of the angular momentum operator in $\mathbb{R}^3$. However, these functions show no peculiar properties under the *general* rotation of SO(4).

Excepted for some cases (lens and prism spaces, see below), the search for the eigenmodes of the spherical spaces of the form S^3/Γ remains an open problem. Since those are eigenmodes of S^3 which remain invariant under the rotations of Γ, it is clear that this search requires an understanding of the rotation properties of the basis functions under SO(4).

The task of this paper is to examine the rotation properties of the eigenfunctions of $\mathcal{V}^k$, as a preparatory work for the search for eigenfunc-

tions of S^3/Γ (in particular for dodecahedral space). This will be done through the introduction of a new basis B3 of $\mathcal{V}^k$ (in the case k even), for which the rotation properties can be explicitely calculated: following a new procedure (that was already applied to S^2 in [5]) we generate a system of coherent states on $\mathcal{V}^k$. We extract from it a basis B3 of $\mathcal{V}^k$, which seems to have been ignored in the literature, and presents original properties. Each function Φ^k_{IJ} of this basis B3 is defined as [the reduction to S^3 of] an homogeneous harmonic polynomial in $\mathbb{R}^4$, which takes the very simple form $(X \cdot N)^k$. Here, the dot product extends the Euclidean [scalar] dot product of $\mathbb{R}^4$ to its complexification $\mathbb{C}^4$, and N is a null vector of $\mathbb{C}^4$, that we specify below. After defining these functions, we show that they form a basis of $\mathcal{V}^k$, and we give the explicit transformation formulae between B2 and B3.

The properties of the basis B3 differ from those of the two other bases, and make it more convenient for particular applications. In particular, it is possible to calculate explicitely its rotation properties, under an arbitrary rotation of SO(4), by using their quaternionic representation (section 3). This allows to find those functions which remain invariant under an arbitrary rotation. In section 4, we apply these result to (re-)derive the eigenmodes of lens and prism space.

2 Harmonic functions

A function f on S^3 is an eigenmode [of the Laplacian] if it verifies $\Delta f = \lambda f$. It is known that eigenvalues are of the form $\lambda_k = -k\ (k+2)$, $k \in \mathbb{N}^+$. The corresponding eigenfunctions generate the eigen[vector]space $\mathcal{V}^k$, of dimension $(k+1)^2$, which realizes an irreducible unitary representation of the group SO(4).

First basis

I call B1 the most widely used basis for $\mathcal{V}^k$ provided by the hyperspherical harmonics

$$B1 \equiv (\mathcal{Y}_{k\ell m} \propto Y_{\ell m}),\ \ell = 1..k,\ m = -\ell..\ell. \tag{1}$$

It generalizes the usual spherical harmonics $Y_{\ell m}$ on the sphere. In fact, it can be shown ([1], [2] p.240,[3]) that a basis of this type exists on any sphere S^n. Moreover, [2] [3] show that the B1 basis for S^n is " naturally generated " by the B1 basis for S^{n-1}. In this sense, the B1 basis for S^3 is generated by the usual spherical harmonics $Y_{\ell m}$ on the 2-sphere S^2.

The generation process involves harmonic polynomials constructed from null complex vectors (see below). The basis B1 is in fact based on the reduction of the representation of SO(4) to representations of SO(3): each $\mathcal{Y}_{k\ell m}$ is an eigenfunction of an SO(3) subgroup of SO(4) which leaves a selected point of S^3 invariant. This make these functions useful when one considers the action of that peculiar SO(3) subgroup. But they show no simple behaviour under a general rotation. We will no more use this basis.

Second basis

By group theoretical arguments, [1] construct a different ON basis of V^k, which is specific to S^3:

$$B2 \equiv (T_{k;m_1,m_2}),\ m_1, m_2 = -k/2...k/2, \tag{2}$$

where m_1 and m_2 vary independently by entire increments (and, thus, take entire or semi-entire values according to the parity of k). In the

spirit of the construction refered above, B2 may be seen as generated from a different choice of spherical harmonics on S^2. The bases B1 and B2 appear respectively adapted to the systems of hyperspherical and toroidal (see below) coordinates to describe S^3.

The formula (27) of [1], reduced to the three-sphere, shows that the elements of this basis take a very convenient form if we use *toroidal coordinates* (as they are called by [7]) on the three sphere S^3: (χ, θ, ϕ) spanning the range $0 \leq \chi \leq \pi/2$, $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq 2\pi$. They are conveniently defined (see [7] for a more complete description) from an isometric embedding of S^3 in $\mathbb{R}^4$ (as the hypersurface $x \in \mathbb{R}^4$; $\mid x \mid = 1$):

$$\begin{cases} x^0 &= r \ \cos\chi \ \cos\theta \\ x^1 &= r \ \sin\chi \ \cos\phi \\ x^2 &= r \ \sin\chi \ \sin\phi \\ x^3 &= r \ \cos\chi \ \sin\theta \end{cases}$$

where (x^μ), $\mu = 0, 1, 2, 3$, is a point of $\mathbb{R}^4$. As shown in [7]), these coordinates appear naturally associated to some isometries.

Very simple manipulations show that, with these coordinates, each eigenfunction of B2 takes the form:

$$T_{k;m_1,m_2}(X) \equiv t_{k;m_1,m_2}(\chi) \ e^{i\ell\theta} \quad e^{im\phi}, \tag{3}$$

where the $t_{k;m_1,m_2}(\chi)$ are polynomials in $\cos\chi$ and $\sin\chi$ and we wrote, for simplification, $\ell \equiv m_1 + m_2$, $m \equiv m_2 - m_1$.

To have a convenient expression, we report this formula in the harmonic equation expressed in coordinates χ, θ, ϕ. This leads to a second order differential equation (cf. equ 15 of [7]). The solution is proportional to a Jacobi polynomial: $t_{k;m_1,m_2}(\chi) \propto \cos^\ell\chi \ \sin^m\chi \ P_d^{m,\ell}(\cos 2\chi)$, $d \equiv k/2 - m_2$. Thus, we have the final expression for the basis B2

$$T_{k;m_1,m_2}(X) = C_{k;m_1,m_2} \ [\cos\chi \ e^{i\theta}]^\ell \ [\sin\chi \ e^{i\phi}]^m \ P_d^{(m,\ell)}[\cos(2\chi)], \tag{4}$$

with $C_{k;m_1,m_2} \equiv \frac{\sqrt{(k+1)}}{\pi} \sqrt{\frac{(k/2+m_2)! \ (k/2-m_2)!}{(k/2+m_1)! \ (k/2-m_1)!}}$ from normalization requirements (the variation ranges of m_1 anf m_2 imply that the quantities under factorial sign are entire and positive). Note also the useful proportionality relations:

$$\cos^\ell\chi \ \sin^m\chi \ P^{(m,\ell)}_{\frac{k-\ell-m)}{2}}(\cos 2\chi) \propto \cos^\ell\chi \ \sin^{-m}\chi \ P^{(-m,\ell)}_{\frac{k-\ell+m)}{2}}(\cos 2\chi)$$
$$\propto \cos^{-\ell}\chi \ \sin^m\chi \ P^{(m,-\ell)}_{\frac{k+\ell-m}{2}}(\cos 2\chi).$$

The term $\zeta^m \ \xi^n \equiv e^{i\ell\theta} \ e^{im\phi}$ in (4) defines the rotation properties of $T_{k;m_1,m_2}$ under a specific subgroup of SO(4). This properties generalizes the properties of the spherical harmonics on the two-sphere S^2, to be eigenfunctions of the rotation operator P_x. This advantage has been used by [7] to calculate (from a slightly different basis) the eigenmodes of lens or prism spaces (see below section 4). However, the $T_{k;m_1,m_2}$ have no simple rotation properties under the general rotation of SO(4). This motivates the search for a different basis of $\mathcal{V}^k$.

Note that the basis functions $T_{k;m_1,m_2}$ have also been introduced in [2] (p. 253), with their expression in Jacobi Polynomials. Note also that they are the complex counterparts of those proposed by [7] (their equ. 19) to find the eigenmodes of lens and prism spaces. The variation range of the indices m_1, m_2 here (equ. 2) is equivalent to their condition

$$\mid \ell \mid + \mid m \mid \leq k, \ \ell + m = k, \quad \bmod (2), \tag{5}$$

through the correspondence $\ell = m_1 + m_2$, $m = m_2 - m_1$.

2.1 Complex null vectors

A complex vector $Z \equiv (Z^0, Z^1, Z^2, Z^3)$ is an element of $\mathbb{C}^4$. We extend the Euclidean scalar product in $\mathbb{R}^4$ to the complex (non Hermitian) inner product $Z \cdot Z' \equiv \sum_\mu Z^\mu \ (Z')^\mu$, $\mu = 0, 1, 2, 3$. A *null* vector N is defined as having zero norm $N \cdot N \equiv \sum_\mu N^\mu N^\mu = 0$ (in which case, it may be considered as a point on the isotropic cone in $\mathbb{C}^4$). It is well known that polynomials of the form $(X \cdot N)^k$, homogeneous of degree k, are harmonic if and only if N is a null vector. This results from

$$\Delta_0 (X \cdot N)^k \equiv \sum_\mu \partial_\mu \ \partial_\mu \ (X \cdot N)^k = k \ \left(\sum_\mu (N_\mu \ N_\mu) \right) \ (X \cdot N)^{k-1} = 0,$$

where Δ_0 is the Laplacian of $\mathbb{R}^4$. Thus, the restrictions of such polynomials belong to $\mathcal{V}^k$. As we mentioned above, peculiar null vectors have been used in [2] and [3] to generate the bases B1 and B2.

To construct a third basis B3, let us first define a family of null vectors

$$N(a, b) \equiv (cosa, \ i \ \sin b, \ i \ \cos b, \ \sin a), \tag{6}$$

indexed by two angles a and b describing the unit circle (they define coherent states in $\mathbb{R}^4$).

The polynomial $[X \cdot N(a, b)]^k$ is harmonic and, thus, can be decomposed onto the basis B2. It is easy to check that, like the scalar product $X \cdot N(a, b)$, this polynomial depends on a and b only through the combinations $e^{i(\theta - a)}$ and $e^{i(\phi + b)}$, with their conjugates. This implies that its decomposition on B2 takes the form

$$[X \cdot N(a, b)]^k = \sum_{m_1, m_2} P_{k; m_1, m_2} \ T_{k; m_1, m_2}(X) \ e^{-ia(m_1 + m_2)} \quad e^{ib(m_2 - m_1)}, \tag{7}$$

where the coefficients $P_{k; m_1, m_2}$ do not depend on a, b. Now we intend to find a basis of $\mathcal{V}^k$ in the form of such polynomials.

2.2 An new basis

2.2.1 Roots of unity

To do so, we consider the $(k+1)^{th}$ complex roots of unity which are the powers ρ^I of

$$\rho \equiv e^{\frac{2i\pi}{k+1}} \equiv \cos\alpha + i \sin\alpha, \ \ \alpha \equiv \frac{2\pi}{k+1}. \tag{8}$$

We recall the fundamental property, which will be widely used thereafter:

$$\sum_{n=0}^{k} \rho^{nI} = (k+1) \ \delta_I^{Dirac}, \tag{9}$$

where the equallity in the Dirac must be taken $\mod k+1$.

In a given frame, we consider the family of null vectors

$$N_{IJ} \equiv N(I\alpha, J\alpha) = (\cos I\alpha, \ i \ \sin J\alpha, \ i \ \cos J\alpha, \ \sin I\alpha), \ \ I, J = 0..k \tag{10}$$

and we define the functions Φ^k_{IJ} : $\Phi^k_{IJ}(X) \equiv (X \cdot N_{IJ})^k$. We report such a function in equ.(7) to obtain its development in the basis B2. Then we multiply both terms by $\rho^{I(m_1+m_2) - J(m_2 - m_1)}$. Making the summations

over I, J (each varying from 0 to k), and using (9), we obtain, in the case where k is even (that we assume hereafter):

$$\mathcal{T}_{k;m_1,m_2} = \frac{1}{(k+1)^2} \sum_{I,J=0}^{k} \rho^{I(m_1+m_2)-J(m_2-m_1)} \Phi^k_{IJ}, \tag{11}$$

where $\mathcal{T}_{k;m_1,m_2} \equiv P_{k;m_1,m_2} T_{k;m_1,m_2}$.

This gives the decomposition of any $T_{k;m_1,m_2}$ (and thus, of any harmonic function) as a sum of the $(k+1)^2$ polynomials Φ^k_{IJ}, providing the new basis of $\mathcal{V}^k$:

$$B3 \equiv (\Phi^k_{IJ}),\ I, J = 0..k \qquad (k \text{ even}). \tag{12}$$

The coefficients $P_{k;m_1,m_2}$ involved in the transformation are calculated in Appendix A. We obtain easily the reciprocal formula expressing the change of basis:

$$\Phi^k_{IJ} = \sum_{m_1,m_2=-k/2}^{k/2} \mathcal{T}_{k;m_1,m_2} \rho^{-I(m_1+m_2)+J(m_2-m_1)}. \tag{13}$$

3 Rotations in $\mathbb{R}^4$

3.1 Matrix representations

The isometries of S^3 are the rotations in the embedding space $\mathbb{R}^4$. In the usual matrix representation, a rotation is represented by a $4*4$ orthogonal matrix $g \in \mathrm{SO}(4)$, acting on the 4-vector (x^μ) by matrix product.

In the complex matrix representation, a point (vector) of $\mathbb{R}^3$ is represented by the $2*2$ complex matrix

$$X \equiv \begin{bmatrix} W & iZ \\ i\bar{Z} & \bar{W} \end{bmatrix};\ W \equiv x^0 + ix^3,\ Z \equiv x^1 + ix^2 \in \mathbb{C}.$$

A rotation g is represented by two complex $2*2$ matrices (G_L, G_R), so that its action takes the form $X \mapsto G_L\ X\ G_R$ (matrix product). The two matrices G_L and G_R belong to SU(2). Since SU(2) identifies with S^3, any matrix G_L or G_R is of the same form than the matrix X above. Since SU(2) is also the set of unit norm quaternions, there is a quaternionic representation for the action of SO(4).

3.2 Quaternionic notations

Let us note $j_\mu,\ \mu = 0, 1, 2, 3$ the basis of quaternions (the j_μ correspond to the usual $1, i, j, k$ but we do not use this notation here). We have $j_0 = 1$. A general quaternion is $q = q^\mu j_\mu = q^0 + q^i j_i$ (with summation convention; the index i takes the values 1,2,3; the index μ takes the values 0,1,2,3). Its quaternionic conjugate is $\bar{q} \equiv q^0 - q^i j_i$. The scalar product is $q_1 \cdot q_2 \equiv (q_1\ \bar{q}_2 + q_2\ \bar{q}_1)/2$, giving the quaternionic norm $\mid q \mid^2 = \frac{q\bar{q}}{2} = \sum_\mu (q^\mu)^2$.

We represent any point $x = (x^\mu)$ of $\mathbb{R}^4$ by the quaternion $q_x \equiv x^\mu\ j_\mu$. The points of the (unit) sphere S^3 correspond to units quaternions, $\mid q \mid^2 = 1$. Hereafter, all quaternions will be unitary (if no otherwise indicated). It is easy to see that, using the coordinates above, a point of S^3 is represented by the quaternion $\cos\chi\ \dot{\zeta} + \sin\chi\ \dot{\xi}\ j_1$, where we define dotted quantities, like $\dot{\zeta} \equiv \cos\theta + j_3\ \sin\theta$, $\dot{\xi} \equiv \cos\phi + j_3\ \sin\phi$, as the quaternionic analogs

of the complex numbers $\zeta = \cos\theta + i\ \sin\theta$ and $\xi = \cos\phi + i\ \sin\phi$, i.e., with the imaginary i replaced by the quaternion j_3.

In quaternionic notation, the rotation $g:\ x \mapsto gx$ is represented by a pair of unit quaternions (Q_L, Q_R) such that $q_x \mapsto q_{gx} = Q_L\ q_x\ Q_R$.

Complex quaternions, null quaternions

The null vectors N introduced above do not belong to $\mathbb{R}^4$ but to $\mathbb{C}^4$. Thus, they cannot be represented by quaternions, but by complex quaternions. Those are defined exactly like the usual quaternions, but with complex rather than real coefficients. Note that the pure imaginary i does not coincide with any of the j_μ, but commutes with all of them. Also, complex conjugation (star) and quaternionic conjugation (bar) must be carefully distinguished. Then it is easy to see that the (null) vectors N_{IJ} defined above correspond to the complex quaternion $n_{IJ} \equiv \dot{\rho}^I + i\ j_2\ \dot{\rho}^J$. Note that $\mid n_{IJ} \mid^2 = 0$.

In quaternionic notations, the basis functions are expressed as

$$\Phi_{IJ}(x) = (N_{IJ} \cdot x)^k = < n_{IJ} \cdot q_X >^k = \Big(\frac{n_{IJ}\ \bar{q_X} + q_X\ \bar{n}_{IJ}}{2} \Big)^k . \qquad (14)$$

Quaternionic notations will help us to check how the basis functions are transformed by the rotations of SO(4).

3.3 Rotations of functions

To any rotation g, is associated its action $\mathbf{R}_g$ on functions: $\mathbf{R}_g : f \mapsto \mathbf{R}_g f$; $\mathbf{R}_g f(x) \equiv f(gx)$. Let us apply this action to the basis functions:

$$\mathcal{R}_g \Phi_{IJ}(x) = \Phi_{IJ}(gx) = < n_{IJ} \cdot (Q_L\ q_x\ Q_R) >^k . \qquad (15)$$

We consider a function on S^3 also as a functions on the set of unit quaternions (q_x is the unit quaternion associated to the point x of S^3). On the other hand, we may develop this function on the basis:

$$\mathbf{R}_g \Phi_{IJ} \equiv \sum_{ij=0}^{k} G_{IJ}^{ij}(g)\ \Phi_{ij}. \qquad (16)$$

The coefficients $G_{IJ}^{ij}(g)$ of the development, that we intend to calculate, completely encode the action of the rotation g on the basis B3, and thus on V^k.

To proceed , we introduce three auxiliary complex quaternions:

$$\alpha \equiv 1 + i\ j_3,\ \beta \equiv j_1 - i\ j_2 = (1 - i\ j_3)\ j_1 \text{ and } \delta \equiv -j_1 - i\ j_2.$$

They have zero norm and obey the properties $< \alpha \cdot n_{IJ} > = \rho^I$, $< \bar{\alpha} \cdot n_{IJ} > = \rho^{-I}$, $< \beta \cdot n_{IJ} > = \rho^J$, $< \delta \cdot n_{IJ} > = \rho^{-J}$. Let us now estimate the relation (16) for the specific quaternion $\alpha + R\ \bar{\alpha} + S\ \beta + T\ \delta$, with R, S, T arbitrary real numbers:

$$(\mathcal{A} + R\,\mathcal{A}' + S\,\mathcal{B} + T\,\mathcal{D})^k = \sum_{ij} G_{IJ}^{ij}(g)\ < (\rho^i + R\,\rho^{-i} + S\,\rho^j + T\,\rho^{-j})^k, \quad (17)$$

where $\mathcal{A} \equiv < Q_L\ \alpha\ Q_R \cdot n_{IJ} >$, $\mathcal{A}' \equiv < Q_L\ \bar{\alpha}\ Q_R \cdot n_{IJ} >$, $\mathcal{B} \equiv < Q_L\ \beta\ Q_R \cdot n_{IJ} >$, $\mathcal{D} \equiv < Q_L\ \delta\ Q_R \cdot n_{IJ} >$ characterize the rotation. (Note that these quantities depend on I and J).

We develop and identify the powers of the exponents R, S, T:

$$\mathcal{A}^q\ \mathcal{A}'^{p-q}\ \mathcal{B}^r\ \mathcal{D}^{k-p-r} = \sum_{ij} G_{IJ}^{ij}(g)\ \rho^{i(2q-p)}\ \rho^{j(2r-k+p)}.$$

This holds for $0 \leq q \leq p,\ 0 \leq r \leq k-p,\ 0 \leq p \leq k$. After definition of the new indices $A \equiv q+r,\ B \equiv q-r+k-p$, which both vary from 0 to k, the previous equation takes the form

$$\left(\frac{\mathcal{A}\mathcal{B}}{\mathcal{A}'\mathcal{D}}\right)^{A/2} \left(\frac{\mathcal{A}\mathcal{D}}{\mathcal{A}'\mathcal{B}}\right)^{B/2} \left(\frac{\mathcal{A}\mathcal{A}'}{\mathcal{B}\mathcal{D}}\right)^{p/2} \left(\frac{\mathcal{A}'\mathcal{B}\mathcal{D}}{\mathcal{A}}\right)^{k/2} = \sum_{ij} G_{IJ}^{ij}(g)\ \rho^{i(A+B-k)+j(A-B)}.$$

This holds for any value of A, B, p. A consequence is that $\mathcal{A}\mathcal{A}' = \mathcal{B}\mathcal{D}$, which can be checked directly. Finally,

$$\mathcal{U}^A \mathcal{V}^B\ \left(\mathcal{A}'\right)^k = \sum_{ij} G_{IJ}^{ij}(g)\ \rho^{i(A+B-k)}\ \rho^{j(A-B)},$$

with $\mathcal{U} \equiv \left(\frac{\mathcal{B}}{\mathcal{A}'}\right)$, $\mathcal{V} \equiv \left(\frac{\mathcal{A}}{\mathcal{B}}\right)$.

Taking into account the properties of the roots of unity, this equation has the solution

$$G_{IJ}^{ij} = \frac{\left(\mathcal{A}'\right)^k}{(k+1)^2} \sum_{A,B=0}^{k} \rho^{-i(A+B-k)}\ \rho^{-j(A-B)}\ \mathcal{U}^A\ \mathcal{V}^B. \tag{18}$$

When a rotation is specified, there is no difficulty to estimate the associated values of $\mathcal{A}'$, $\mathcal{U}$, $\mathcal{V}$, and thus of these coefficients which completely encode the transformation properties of the basis functions of V^k under SO(4).

In the next section, we apply these results to rederive the eigenmodes of Lens or Prism space. In the next paper [6], we take for g the generators of Γ, the group of holonomies of the dodecahedral space. This will allow the selection of the invariant functions, which constitute its eigenmodes.

4 Lens and Prism space

The eigen modes for Lens and Prism space have been found by [7]. Here we derive them again for illustration of our method.

4.1 Lens space

An holonomy transformation of a lens space takes the form, in complex notation,

$$G_L = \begin{bmatrix} e^{i\frac{\psi_1+\psi_2}{2}} & 0 \\ 0 & e^{-i\frac{\psi_1+\psi_2}{2}} \end{bmatrix},\ G_R = \begin{bmatrix} e^{i\frac{\psi_1-\psi_2}{2}} & 0 \\ 0 & e^{-i\frac{\psi_1-\psi_2)}{2}} \end{bmatrix} G. \tag{19}$$

Its action on a vector of $\mathbb{R}^4$ takes the form

$$X \equiv \begin{bmatrix} W & iZ \\ i\bar{Z} & \bar{W} \end{bmatrix} \mapsto G_L\ X\ G_R = \begin{bmatrix} We^{i\psi_1} & iZe^{i\psi_2} \\ i\bar{Z}e^{-i\psi_2} & \bar{W}e^{-i\psi_1} \end{bmatrix}. \tag{20}$$

In this simple case, $W \equiv x^0 + ix^3$, $Z \equiv x^1 + ix^2$ are transformed into $W\ e^{i\psi_1}$ and $Z\ e^{i\psi_2}$ respectively. This corresponds to the quaternionic notation

$$Q_L = \dot{w}_1\ \dot{w}_2,\ Q_R = \dot{w}_1/\dot{w}_2,\ \dot{w}_i \equiv \cos(\psi_i/2) + j_3\ \sin(\psi_i/2). \tag{21}$$

The rotation is expressed in the simplest way in the toroidal coordinates, since it acts as $\theta \mapsto \theta + \psi_1,\ \phi \mapsto \phi + \psi_2$. From the expression (4) of the basis functions (B2), it result their transformation law :

$$\mathbf{R}_g : T_{k;m_1,m_2} \mapsto T_{k;m_1,m_2}\ e^{\ell\psi_1 + m\psi_2}.$$

This leads directly to the invariance condition $\ell\psi_1 + m\psi_2 = 0 \mod 2\pi$. Using the standard notation for a lens space $L(p,q)$, namely

$$\psi_1 = 2\pi/p,\ \ \psi_2 = 2\pi\ q/p,$$

we are led to the conclusion:

> the eigenmodes of lens space $L(p,q)$ are all linear combinations of $T_{k;\underline{m_1},\underline{m_2}}$, where the underlining means that the indices verify the condition $\underline{m_1} + \underline{m_2} + q(\underline{m_2} - \underline{m_1}) = 0$, modulo$(p)$.

4.2 Prism space

The two generators are single action rotations ($G_R = 0$). The first generator, analog to the lens case above, with $\psi_1 = \psi_2 = 2\pi/2P$, provides the first condition $\ell + m = 0, \quad \mod 2P$ which takes the form

$$\underline{m_2} = 0 \quad \mod P. \tag{22}$$

This implies that k must be even.

The second generator has the complex matrix form $G = G_L = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$, which corresponds to the quaternion $Q_L = Q = -j_1$. Easy calculations lead to $\mathcal{A} = -\rho^J$, $\mathcal{A}' = \rho^{-J}$, $\mathcal{B} = \rho^I$, $\mathcal{D} = -\rho^{-I}$. Reporting in (18) gives

$$G_{IJ}^{ij} = \frac{\rho^{(i-J)\ k}}{(k+1)^2} \sum_{A,B=0}^{k} \rho^{A\ (-i-j+I+J)+B\ (-i+j-I+J)}\ (-1)^B. \tag{23}$$

This formula, together with those expressing the change of basis between B2 and B3, allow to return to the rotation properties of the basis B2 which take the simple form:

$$\mathbf{R} : \mathcal{T}_{k;m_1,m_2} \mapsto (-1)^{m_2+k/2}\ \mathcal{T}_{k;m_1,-m_2}. \tag{24}$$

It results immediately that the G-invariant functions are combinations of $\mathcal{T}_{k;m_1,m_2} + (-1)^{m_2+k/2}\ \mathcal{T}_{k;m_1,-m_2}$.

Finally,

> the eigenfunctions of the Prism space are combinations of $\mathcal{T}_{k;m_1,\underline{m_2}} + (-1)^{\underline{m_2}+k/2}\ \mathcal{T}_{k;m_1,-\underline{m_2}},\ \forall m_1;\ \ k$ even.

According to the parity of $k/2$, the functions $\mathcal{T}_{k;m_1,0}$ are included or not, from which simple counting give the multiplicity as
$(k+1)\ (1+[k/2P])$, for k even ([...] means entire value),
$(k+1)\ [k/2P]$, for k odd, in accordance with [4].

5 Conclusion

We have shown that V^k, the space of eigenfunctions of the Laplacian of S^3 with a given eigenvalue λ_k (k even) admits a new basis B3. In contrary to standard bases (B1 and B2) which show specific rotation properties under selected subgroups of SO(4), it is possible to calculate explicitely the rotation properties of B3 under any rotation of the group SO(4), as well as to calculate the functions invariant under this rotation. This opens the door to the calculation of eigenmodes of spherical space. The

eigenfunctions of lens and prism spaces had been calculated by [7], by using a basis related to B2 (its real, rather than complex, version). We rederived them to illustrate the properties of the bases.

In a subsequent paper [6], we apply these results to the search of the eigenfunctions of the dodecahedral space S^3/Γ, where $\Gamma = D_P^*$ is the binary dihedral group of order $4P$. Those functions, still presently unknown, are the eigenfunctions of S^3 which remain invariant under the elements of Γ.

5.1 Appendix A

Let us evaluate the function

$$Z_{\ell m}^k(X) \equiv \sum_{IJ=0}^{k} \rho^{\ell I - Jm} \ \Phi_{IJ}^k(X) \tag{25}$$

$$= 2^{-k} \sum_{IJ} \rho^{\ell I - mJ} \left[\cos\chi \ (\zeta\rho^{-I} + \frac{1}{\zeta\rho^{-I}}) + \sin\chi \ (\xi\rho^{J} - \frac{1}{\xi\rho^{J}})\right]^k,$$

where we defined $\zeta \equiv e^{i\theta}$ and $\xi \equiv e^{i\phi}$. After development of the power with the binomial coefficients, the sum becomes

$$\sum_{IJ} \rho^{\ell I - mJ} \sum_{p=0}^{k} \left(\begin{array}{c} k \\ p \end{array} \right) [\cos\chi \ (\zeta\rho^{-I} + \frac{1}{\zeta\rho^{-I}})]^{k-p} \ [\sin\chi \ (\xi\rho^{J} - \frac{1}{\xi\rho^{J}})]^p. \tag{26}$$

Let us write the identities

$$\rho^{\ell I} \ (\zeta\rho^{-I} + \frac{1}{\zeta\rho^{-I}})^{k-p} = \sum_{r=0}^{k-p} \left(\begin{array}{c} k-p \\ r \end{array} \right) \zeta^{2r+p-k} \ \rho^{-I(2r+p-k-\ell)}, \tag{27}$$

$$\rho^{-mJ} \ (\xi\rho^{J} - \frac{1}{\xi\rho^{J}})^{p} = \sum_{q=0}^{p} \left(\begin{array}{c} p \\ q \end{array} \right) \xi^{2q-p} (-1)^{p-q} \ \rho^{J(2q-p-m)}, \tag{28}$$

that we insert into (26). After summing over I, J, and rearranging the terms, we obtain:

$$Z_{\ell m}(X) = 2^{-k} \ \zeta^{\ell} \ \xi^{m} \ k! \sum_{q}^{k} \frac{(-1)^{q-m} \ (\cos\chi)^{k-2q+m} \ (\sin\chi)^{2q-m}}{q! \ (q-m)! \ (\frac{k+\ell-2q+m}{2})! \ (\frac{k-\ell-2q+m}{2})!}. \tag{29}$$

This formula results from the fact that, through (9), the summations over I, J imply $p = 2q - m$ and $2r = \ell + k + m - 2q$, that we have reported. The range of the summation over q is defined by the conditions

$$0 \leq \ell + k + m - 2q \leq 2k + 2m - 4q \leq 2k, \quad 0 \leq q \leq 2q - m \leq k. \tag{30}$$

Rearrangements of the previous formula, inserting $u \equiv \cos(2\chi) = 2\cos^2\chi - 1 = 1 - 2\sin^2\chi$, lead to

$$Z_{\ell m}(X) = \frac{2^{-3k/2} \ \zeta^{\ell} \ \xi^{m} \ k! \ (1+u)^{\frac{\ell}{2}} \ \ (1-u)^{\frac{m}{2}}}{(m+d)! \ (\ell+d)!} \tag{31}$$

$$\sum_{q} \left(\begin{array}{c} m+d \\ i \end{array} \right) \left(\begin{array}{c} \ell+d \\ d-i \end{array} \right) (1+u)^{i} \ (u-1)^{d-i},$$

where we have defined $i \equiv \frac{k+m-\ell}{2} - q$ and $d \equiv \frac{k-\ell-m}{2}$. Verification shows that the range defined as above gives exactly the development formula for the Jacobi polynomial. The comparison with (11) gives the coefficient

$$P_{k;m_1,m_2} = \frac{2^{-k}\ k!}{(k/2-m_1)!\ (k/2+m_1)!\ (k+1)^2\ C_{k;m_1,m_2}}$$

$$= \frac{2^{-k}\ \ \pi\ k!\ (k+1)^{-5/2}}{\sqrt{(k/2+m_2)!\ (k/2-m_2)!(k/2+m_1)!\ (k/2-m_1)!}}$$

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